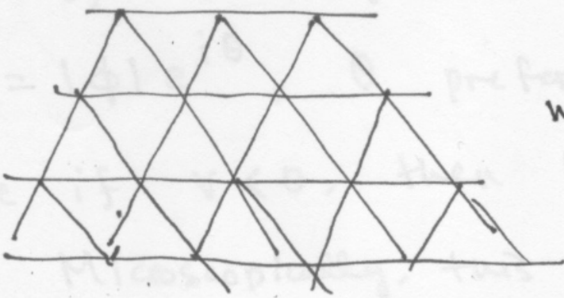
More on Triangular lattice AFM Ising Model



Consider the AMF Ising model on Δ-lattice:

$$H = \sum_{\langle ij \rangle} S_i S_j$$

As you will show in the pset-2, the low energy theory for this model is given by,

$$S = \int d^2x \, |\nabla\phi|^2 + r\,|\phi|^2 + u\,|\phi|^4 + v\,[\phi^6 + \phi^{*6}]$$

where ϕ is a two component complex scalar.

The relation between the microscopic S_i and ϕ is:

$$S(r) = \cos(\vec{Q}\cdot\vec{r})\,\varphi_1(r) + \sin(Q\cdot r)\,\varphi_2(r)$$

where $\phi = \varphi_1 + i\varphi_2$ is slowly fluctuating, and $Q = \frac{4\pi}{3}\hat{x}$.

The implication of the low energy theory is as follows:

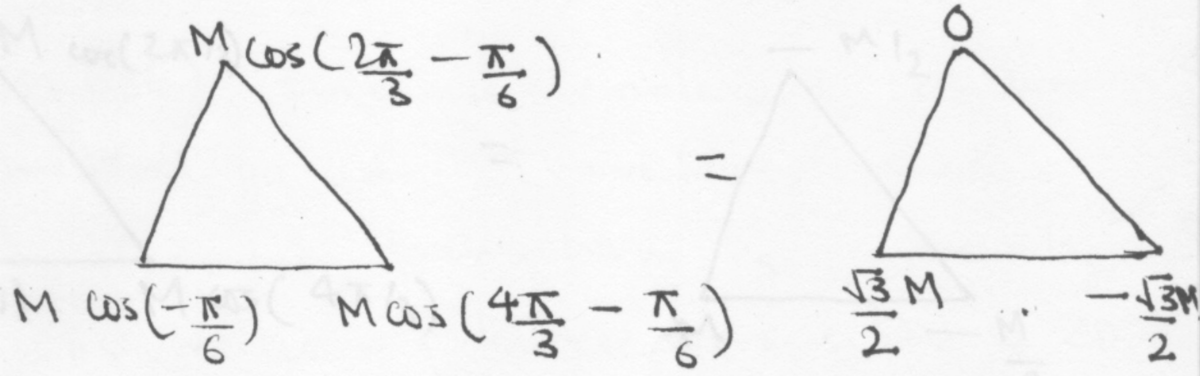
In the low-temperature, sym. broken phase, the ground state will be 6-fold degenerate.

The structure of sym. breaking depends on the sign of r. If $r > 0$, then writing $\phi = |\phi| e^{i\theta}$, θ prefers $\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \ldots$ etc. while if $r < 0$, then θ prefers $\theta = 0, \frac{2\pi}{6}, \ldots$ etc. Microscopically, this translates as follows, when $\theta = \frac{\pi}{6}$, e.g., then

$$S(\mathbf{r}) \propto \cos(\mathbf{Q}\cdot\mathbf{r}) \cos(\theta) + \sin(\mathbf{Q}\cdot\mathbf{r}) \sin\theta$$

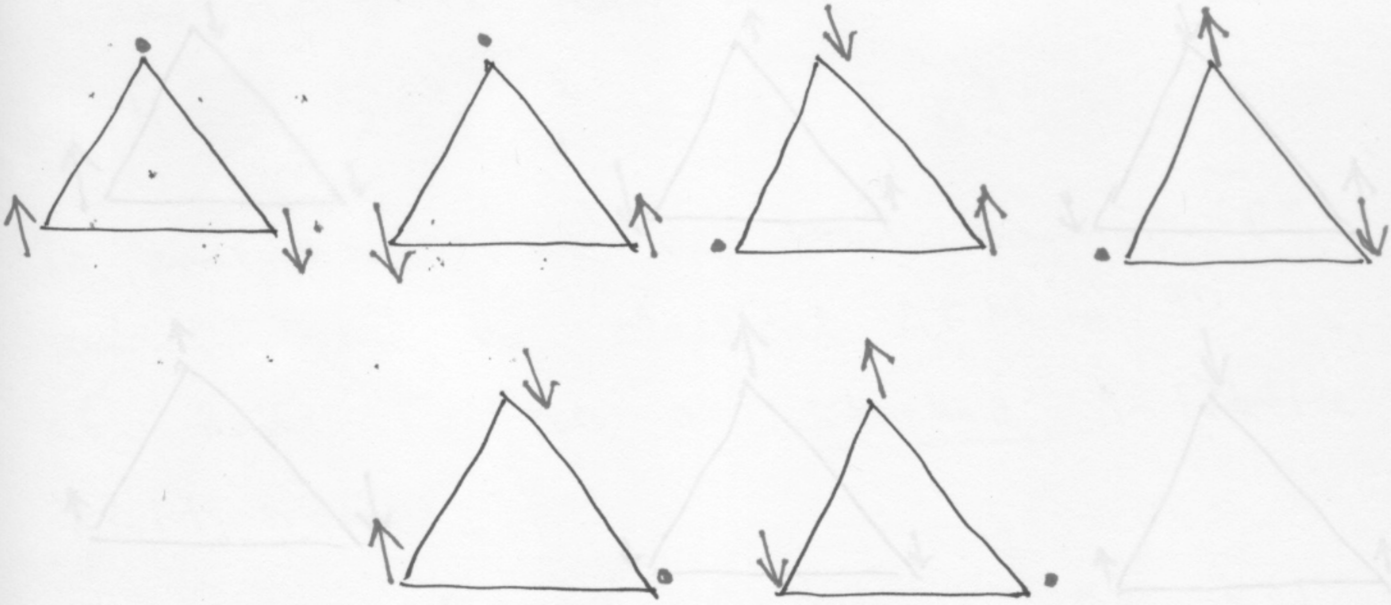
$$= \cos(\mathbf{Q}\cdot\mathbf{r} - \theta).$$

Choosing a triangle where the lower-left corner corresponds to $\mathbf{r} = 0$, the sym. breaking config looks like:



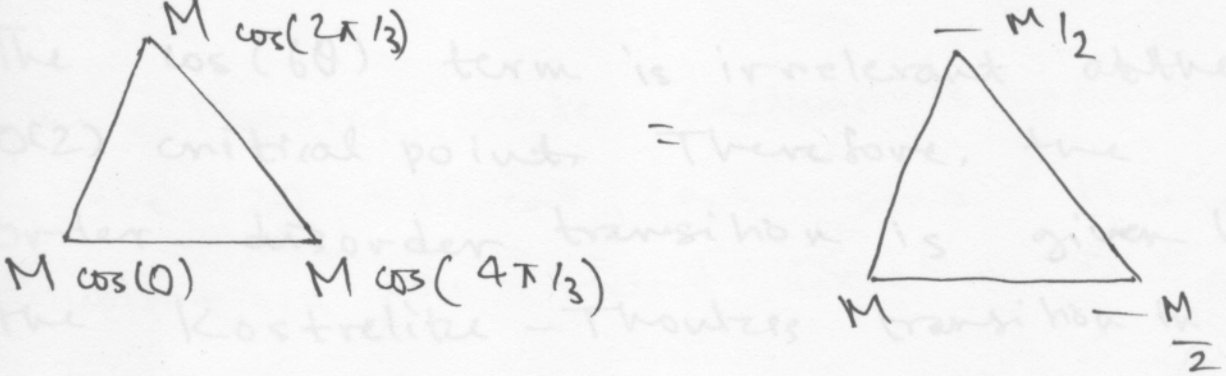
where M is the magnitude $|\phi|$ of the order parameter.

Thus, the six ground states look like:

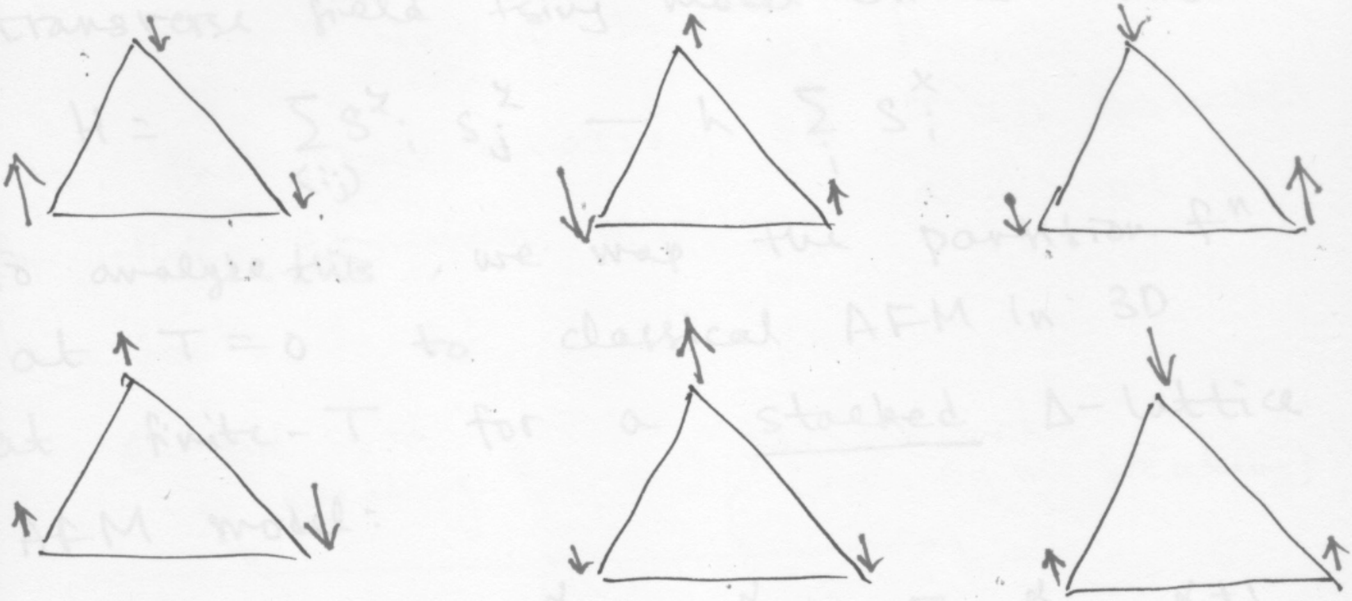


Where the pattern on the rest of the lattice repeats accordingly.

Similarly, when $v < 0$, the sym. breaking config looks like:



Thus the six ground states look like:

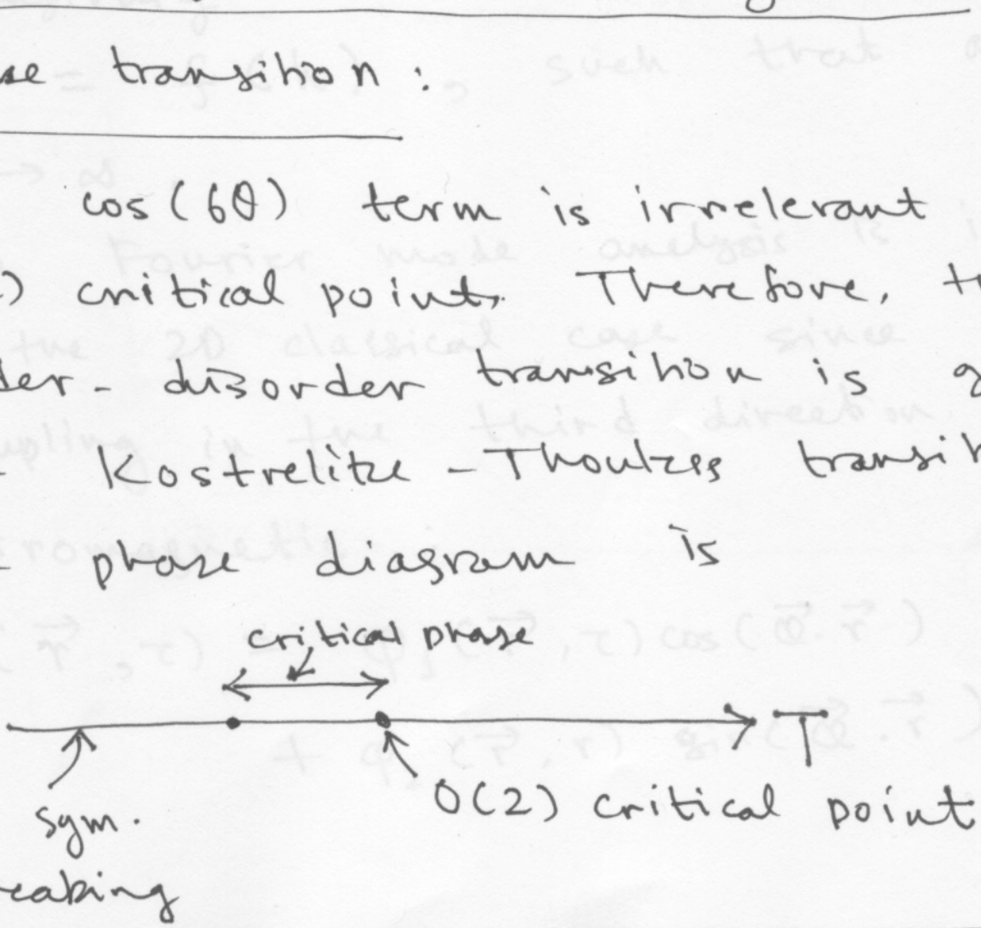


where the larger arrow is twize as big as the smaller arrow.

Implications of field theory for the phase transition:

The $\cos(6\theta)$ term is irrelevant at the O(2) critical point. Therefore, the order-disorder transition is given by the Kostrelitz-Thoulzes transition in 2D.

The phase diagram is



The same analysis applies to quantum transverse field Ising model on Δ-lattice.

$$H = \sum_{\langle ij \rangle} S^z_i S^z_j - h \sum_i S^x_i$$

To analyze this, we map the partition f^n at $T=0$ to classical AFM in 3D at finite-T for a <u>stacked</u> Δ-lattice AFM model:

$$H_{cl} = \sum_{\langle ij \rangle} S^\alpha_i S^\alpha_j - \sum_\alpha S^\alpha_i S^{\alpha+1}_i$$

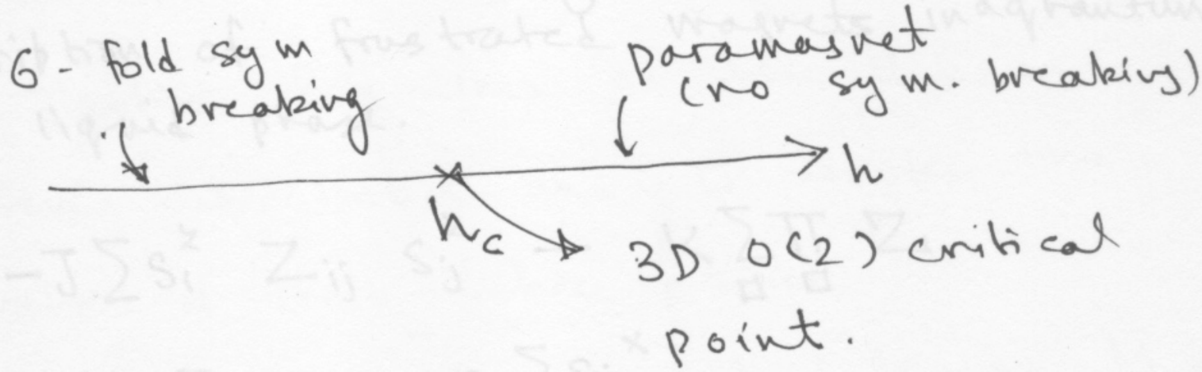
where α is the layer index in the imaginary time direction. The temperature $T = f(h)$, such that as $h \to \infty$, $T \to \infty$.

The Fourier mode analysis is identical to the 2D classical case since the coupling in the third direction is ferromagnetic:

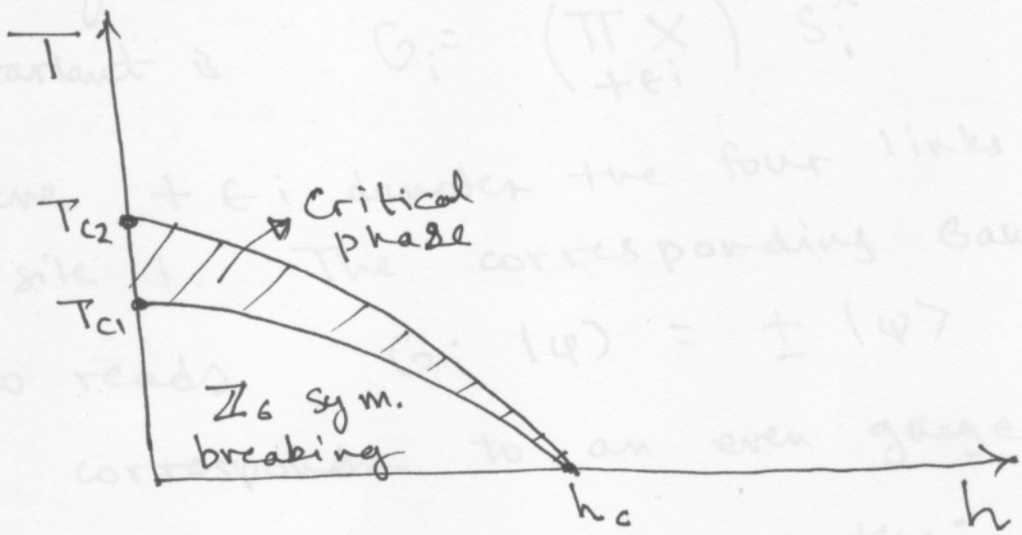
$$S(\vec{r}, \tau) = \varphi_1(\vec{r}, \tau) \cos(\vec{Q} \cdot \vec{r}) + \varphi_2(\vec{r}, \tau) \sin(\vec{Q} \cdot \vec{r})$$

Action $S = \int d^2r\, d\tau \left[|\nabla\phi|^2 + r|\phi|^2 + u|\phi|^4 + v(\phi^6 + \phi^{*6}) \right]$.

Thus, the phase diagram at $T = 0$ ~~looks like~~ for the quantum model as a f^n of h looks like:



Combining everything, the T–h phase digram of the quantum Hamiltonian is:



Phase Diagram of Matter-Gauge Theories

Earlier we studied pure guage theory (e.g. Toric code). Let's study $\mathbb{Z}_2$ guage theory coupled to $\mathbb{Z}_2$ matter field. Such theories arise naturally as low-energy description of frustrated magnets in a quantum spin liquid phase.

$$H = -J \sum S_i^z \; Z_{ij} \; S_j^z - K \sum_{\square} \prod_{\square} Z$$

$$- h \sum X - r \sum_i S_i^x$$

The guage transformation that leaves H invariant is $\quad G_i = \left(\prod_{+\in i} X \right) S_i^x$

where $+ \in i$ denotes the four links connected to site i. The corresponding Gauss law reads $\quad G_i \, |\psi\rangle = \pm |\psi\rangle$ where $+$ corresponds to an even guage theory and $-$ to an odd guage theory.

To analyze the phase diagram, let's map the quantum Hamiltonian in 2D to a classical Hamiltonian in 3D.

$$H_{cl} = -J \sum_{\langle ij \rangle} s_i Z_{ij} s_j - K \sum_{\square} \prod_{\square} Z$$

where s_i lives on all vertices of a cubic lattice, Z_{ij} lives on the links of the same cubic lattice and $\square$ denotes the plaquettes of that cubic lattice.

The guage transformation that leaves H unchanged is:

$$s_i \rightarrow s_i t_i$$

$$Z_{ij} \rightarrow Z_{ij} t_i t_j \quad \text{where } t_i = \pm 1$$

let's chose $t_i = s_i$, so that s_i drop out.

$$H_{cl} = -J \sum_{\langle ij \rangle} Z_{ij} - K \sum_{\square} \prod_{\square} Z$$

When $J=0$, the phase diagram as a fn of K/T (where T = temperature) is same as that of a pure guage theory.

$J=0$:

confined $\longrightarrow$ deconfined

K/T

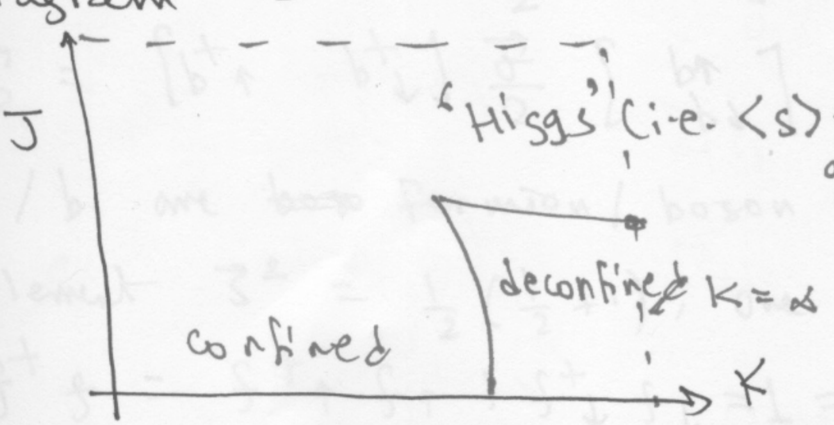
When $K=0$, all links decouple, and there is only a confined phase.

when $J=\infty$ $Z_{ij}=1 \Rightarrow$ confined phase. When $K=\infty$, $\prod_{\square} Z=1$

$\Rightarrow Z = s_i s_j \Rightarrow H_{cl} = -J \sum_{\langle ij \rangle} s_i s_j$

$\Rightarrow$ as a fn of T, order-disorder transition in the Ising universality.

Combining these observations, the phase diagram is:



Higgs and confined are smoothly connected.